Entanglement and spin squeezing properties for three bosons in two modes

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We discuss the canonical form for a pure state of three identical bosons in two modes, and classify its entanglement correlation into two types, the analogous GHZ and the W types as well known in a system of three distinguishable qubits. We have performed a detailed study of two important entanglement measures for such a system, the concurrence $\mathcal{C}$ and the triple entanglement measure τ. We have also calculated explicitly the spin squeezing parameter ξ and the result shows that the W state is the most "anti-squeezing" state, for which the spin squeezing parameter cannot be regarded as an entanglement measure.

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I. INTRODUCTION

Quantum entanglement is an intriguing property of composite systems. It refers to the inseparable correlations stronger than all classical counterparts. Recent studies indicate that entanglement is not only of interest to the interpretation of the foundations of quantum mechanics, but also represents a useful resource for quantum computation and quantum communication. Inseparable correlations such as entanglement also exist in systems of identical particles, e.g. electrons in quantum dots [1], atoms in a Bose-Einstein condensate [2, 3], and electrons in quantum Hall liquids [4]. Even for the widely used parametric down conversion process, a complete treatment must take into account the indistinguishability of the down converted photons. Although the study of entanglement has had a long history in systems of distinguishable particles, only recently did the entanglement properties in a system of identical particles begin to attract much attention [1, 5, 6, 7, 8, 9, 10].

Quantum correlation among identical particles was noted by Schliemann *et al.* [1, 5] and they discussed the entanglement in a two-fermion system. It was argued that the separability of a two fermion state should be defined in terms of whether or not that state can be expressed in terms of a sum of single Slater determinants [1]. More generally, a two fermion pure state can always be expressed in the following standard form [5]

$$|\Psi\rangle = \frac{1}{\sqrt{\sum_{i=1}^{k}|z_i|^2}}\sum_{i=1}^{k} z_i f^\dagger_{a_1(i)} f^\dagger_{a_2(i)}|0\rangle, \qquad (1)$$

where $f^\dagger_{a_1(i)}|0\rangle$ and $f^\dagger_{a_2(i)}|0\rangle$ represent the orthonormal basis states of a single particle (fermion).

The case of two bosons were considered independently by Paškauskas and You [6] and Li *et al.* [7]. They found a similar standard form for two-identical bosons, namely, the wave functions of two bosons can always be written in the following form

$$|\Psi\rangle = \sum_{i=1}^{M} \lambda_i a_i^\dagger a_i^\dagger |0\rangle, \qquad (2)$$

where $a_i^\dagger|0\rangle$ forms an orthonormal basis in the single particle (boson) space.

The above two results are in fact simple extensions of the well known result for two distinguishable particles; that an arbitrary pure state can be described in terms of the famous Schmidt decomposition

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}|i_1 i_2\rangle, \qquad (3)$$

with real parameter λ_i satisfying $\sum_i \lambda_i = 1$ and $\langle i_m|j_n\rangle = \delta_{mn}\delta_{ij}$. A natural generalization of the Schmidt decomposition to $N > 2$ particles is

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}|i_1 i_2 \cdots i_N\rangle, \qquad (4)$$

with the sub-indices m (n) for the m-th (n-th) particle, and i (j) the i-th (j-th) basis vector. However, even for three two-state particles or three qubits with the Hilbert space $\mathcal{H} = C^2 \otimes C^2 \otimes C^2$, the above Schmidt decomposition does not exist, pointing to a truly challenging prospect for characterizing the multi-particle entanglement.

For a multi-particle system, the characterization of its entanglement for a pure state usually starts with certain canonical form of its wave function. In this study, we construct the standard form of an arbitrary wave function for three two-state identical bosons. We further characterize its correlation and squeezing properties based on the standard form. This paper is organized as follows. In Sec. II, we survey the important results on the standard form of a pure state wave function of three distinguishable qubits. We then present our result for the case of three bosons in two modes in Sec. III, which is followed by a detailed discussion of entanglement types,

entanglement measures, and spin squeezing properties, respectively in Sec. IV, Sec. V, and Sec. VI. Finally we discuss the relationship between spin squeezing and pairwise entanglement in our system and conclude with a summary.

II. THE STANDARD FORM OF AN ARBITRARY PURE STATE FOR THREE QUBITS

The Schmidt decomposition does not exist for a three particle pure state, as proven by A. Peres some time ago [12]. We now know that at least five nonlocal parameters are needed to completely specify the LU equivalent types of three qubits [14]. For example as was done by Linden *et al.* [14] using the method of group theory, the following standard form for three qubits can be derived:

$$\begin{aligned} |\psi\rangle_{ABC} &= \sqrt{\lambda}\,|0\rangle\left(a|00\rangle+\sqrt{1-a^2}|11\rangle\right) \\ &\quad +\sqrt{1-\lambda}\,|1\rangle\Big[\gamma\left(\sqrt{1-a^2}|00\rangle-a|11\rangle\right) \\ &\quad +f|01\rangle+g|10\rangle\Big], \end{aligned} \tag{5}$$

where a and f are real numbers and $\gamma=(1-f^2-|g|^2)^{1/2}$. We note this form is a superposition of six orthonormal basis states. In fact, it is found that at least five product orthonormal basis states are needed to express an arbitrary state of three qubits. This is called the generalized Schmidt form of the canonical form by Acin *et al.* [11], later Acin *et al.* [15] further discovered the following "least representation", a superposition of five orthonormal basis states

$$\lambda_0|000\rangle+\lambda_1 e^{i\phi}|100\rangle+\lambda_2|101\rangle+\lambda_3|110\rangle+\lambda_4|111\rangle, \tag{6}$$

where $0\le\phi\le\pi$. This form is also called the "generalized Schmidt decomposition" [15].

If the LU equivalence is used to characterize a three qubit system, infinitely many different types of entanglement are needed due to the different values of the five parameters in Eq. (6). To reduce the entanglement types, Bennett *et al.* [13] introduced the concept of SLOCC (stochastic LOCC) reducible. Dur *et al.* [16] further found that there is only two SLOCC inequivalent types of entanglement for three qubits [16]: the GHZ type

$$|\mathrm{GHZ}\rangle=\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle), \tag{7}$$

and the W type

$$|W\rangle=\frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle), \tag{8}$$

in contrast to what was known earlier for a system of two parties, where only one type of entanglement, *i.e.* the EPR type [17], characterizes all entangled states.

III. THE STANDARD FORM OF AN ARBITRARY PURE STATE FOR THREE TWO-STATE BOSONS

For three identical two-state bosons and in the first quantization representation, the general form of its wave function reads

$$\begin{aligned} |\psi\rangle &= a|000\rangle+b(|100\rangle+|010\rangle+|001\rangle) \\ &\quad +c(|011\rangle+|101\rangle+|110\rangle)+d|111\rangle. \end{aligned} \tag{9}$$

After a single particle basis transformation

$$\begin{aligned} |0\rangle &\rightarrow \alpha|0\rangle+\beta|1\rangle, \\ |1\rangle &\rightarrow -\beta^*|0\rangle+\alpha^*|1\rangle, \end{aligned} \tag{10}$$

the coefficients of transformed basis $|000\rangle$, $|011\rangle$, $|100\rangle$, and $|111\rangle$ become

$$\begin{aligned} |000\rangle &: a\alpha^3-d\beta^{*3}-3b\beta^*\alpha^2+3c\alpha\beta^{*2}, \\ |111\rangle &: a\beta^3+d\alpha^{*3}+3b\alpha^*\beta^2+3c\beta\alpha^{*2}, \\ |011\rangle &: a\alpha\beta^2-d\alpha^*\beta^{*2}-b\beta^*\beta^2+c\alpha\alpha^{*2} \\ &\quad +2b\alpha\alpha^*\beta-2c\beta\beta^*\alpha^*, \\ |100\rangle &: a\beta\alpha^2+d\alpha^*\beta^{*2}+b\alpha^*\alpha^2+c\beta\beta^{*2} \\ &\quad -2b\beta\beta^*\alpha-2c\alpha^*\alpha\beta^*, \end{aligned} \tag{11}$$

As proven in Appendix A, we find that the following proposition holds.

Proposition 1: By properly choosing α and β we can make any one of the above four coefficients zero.

Proposition 1 leads to the following direct corollary with properly chosen phase factors for $|0\rangle$ and $|1\rangle$:

Corollary 1: The wave function of three identical bosons in two modes can be written in the standard form

$$|\psi\rangle=r|000\rangle+s(|100\rangle+|010\rangle+|001\rangle)+t|111\rangle, \tag{12}$$

with r and t real.

Our results in the next three sections will be based on this standard form.

IV. ENTANGLEMENT TYPES

As discussed earlier three distinguishable qubits can be entangled in two different ways [16], denoted by a pure state wave function of the GHZ or the W type. For three identical bosons, we give similar definitions for the two different types as in the following.

Definition 1: Three two-state bosons are GHZ type entangled if its wave function can be written as

$$|\psi\rangle=|\alpha\alpha\alpha\rangle+|\beta\beta\beta\rangle, \tag{13}$$

under appropriate single particle transformations, where $|\alpha\rangle$ and $|\beta\rangle$ are linear independent but need not be orthogonal and orthonormal.

Definition 2: Three two-state bosons are W type entangled iff the wave function can be written as

$$|\psi\rangle = |\alpha\beta\beta\rangle + |\beta\alpha\beta\rangle + |\beta\beta\alpha\rangle \tag{14}$$

under appropriate single particle transformations. $|\alpha\rangle$ and $|\beta\rangle$ are linear independent but need not be orthogonal and orthonormal.

Using proposition 1, it is straightforward to prove **Proposition 2** (see Appendix B): When written in the standard form of Eq. (12), three two-state bosons are GHZ type entangled iff ($r \neq 0$, $t \neq 0$, and $s = 0$), or ($r = 0$, $t \neq 0$, and $s \neq 0$), or ($r \neq 0$, $t \neq 0$, and $s \neq 0$); and they are W type entangled iff ($r = 0$, $t = 0$, and $s \neq 0$), or ($r \neq 0$, $t = 0$, and $s \neq 0$).

It is interesting to note that the parameters r and t are not symmetric with interchange to the middle term in the standard form (12). This observation is consistent with our proposition that the state is W type entangled iff $s \neq 0$ and $t = 0$. This point can be understood intuitively as the basis $|000\rangle$ contains two $|0\rangle$s, and is closer to the W state defined here than the basis $|111\rangle$, thus it is reasonable that only the state $t = 0$ is W entangled.

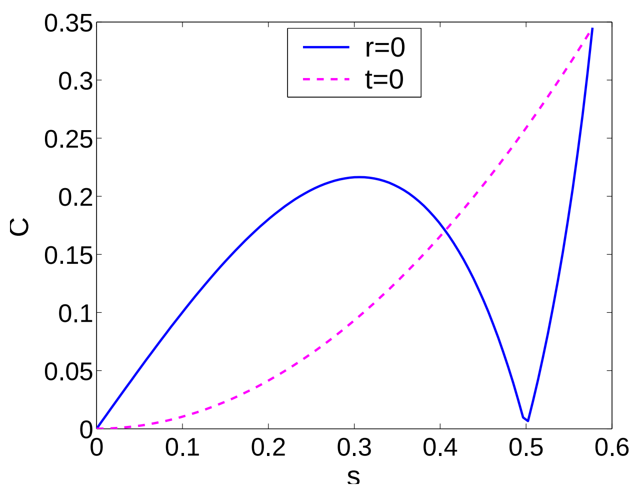

FIG. 1: The concurrences for a pure state of three bosons $|\psi\rangle$ as in Eq. (18) for r=0 (blue solid line) or t=0 (magenta dashed line).

V. ENTANGLEMENT MEASURES

The next task is to measure the entanglement of an arbitrary pure state of three bosons in two modes. As is well known, the concurrence $\mathcal{C}$ and the quantity τ, introduced by Wootters *et al.* [18, 19] are used to measure pairwise and ternary entanglement for two and three qubits respectively. Here we will discuss these entanglement measures for our system of three two-state bosons.

Let us first review the definitions of the concurrence $\mathcal{C}$ and the quantity τ for an arbitrary pure state of three qubits A, B, and C. The concurrence $\mathcal{C}_{AB}$ is defined as

$$\mathcal{C}_{AB} = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \tag{15}$$

where λ_1, λ_2, λ_3, and λ_4 are the square roots of the eigenvalues, in decreasing order, of the following operator

$$\rho_{AB}(\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y), \tag{16}$$

with ρ_{AB} the reduced density matrix of qubits A and B. Similarly, one can define the concurrences $\mathcal{C}_{BC}$, $\mathcal{C}_{AC}$.

Wootters *et al.* [19] found that

$$\begin{aligned} \tau_{ABC} &:= \mathcal{C}^2_{A(BC)} - \mathcal{C}^2_{AB} - \mathcal{C}^2_{AC} \\ &= \mathcal{C}^2_{B(AC)} - \mathcal{C}^2_{AB} - \mathcal{C}^2_{BC} \\ &= \mathcal{C}^2_{C(AB)} - \mathcal{C}^2_{AC} - \mathcal{C}^2_{BC}, \end{aligned} \tag{17}$$

where $\mathcal{C}_{A(BC)}$, $\mathcal{C}_{B(AC)}$, and $\mathcal{C}_{C(AB)}$ are concurrences of the pure state $|\psi\rangle_{ABC}$ with bipartite partitions $A(BC)$, $B(AC)$ and $C(AB)$.

Before presenting our results on entanglement for a three boson pure state, we note that although $\mathcal{C}$ and τ have been customarily used for three distinguishable particles [18, 19], they remain valid for the case of three bosons. This is so because when we construct the decomposition of the two-qubit density matrix ρ that adopts the minimum average pre-concurrence $\mathcal{C}$ (and hence the minimal concurrence of ρ), we start from the eigenvalue decomposition of ρ [18], which is automatically symmetrized for a three-boson system. The quantity τ as defined is also automatically invariant under exchange of particles.

We now calculate from the standard form (12) for three bosons. For convenience, we rewrite Eq. (12) as

$$|\psi\rangle = r|000\rangle + se^{i\phi}(|100\rangle + |010\rangle + |001\rangle) + t|111\rangle, \tag{18}$$

where r, s, t, and ϕ are all real with three of them being independent due to normalization. A direct calculation leads to the following results

$$\begin{aligned} \mathcal{C} &= \sqrt{4t^2s^2 + 2t^2r^2 + 4s^4 + 2\sqrt{t^4s^4 + t^4r^2s^2 - 2s^6t^2 + s^4t^2r^2 + s^8 + 2r^2s^3t^3\cos(3\phi)}} \\ &- \sqrt{4t^2s^2 + 2t^2r^2 + 4s^4 - 2\sqrt{t^4s^4 + t^4r^2s^2 - 2s^6t^2 + s^4t^2r^2 + s^8 + 2r^2s^3t^3\cos(3\phi)}}, \end{aligned} \tag{19}$$

and

$$\tau = 4|r^2t^2 + 4ts^3e^{i3\phi}|, \tag{20}$$

where the reduced two party density matrices are identical $\rho_{AB} = \rho_{BC} = \rho_{AC}$, and can be evaluated in the single particle basis directly, or more generally from the two particle reduced density matrix of a general many body system $\sim \rho^{(2)}_{ijkl} = \langle a_i^\dagger a_j^\dagger a_k a_l \rangle/2!$. When $s = 0$, we find $\mathcal{C} = 0$, i.e. there is no pairwise entanglement in the state $r|000\rangle + t|111\rangle$. Nevertheless, there exists ternary entanglement $\tau = 4r^2t^2$.

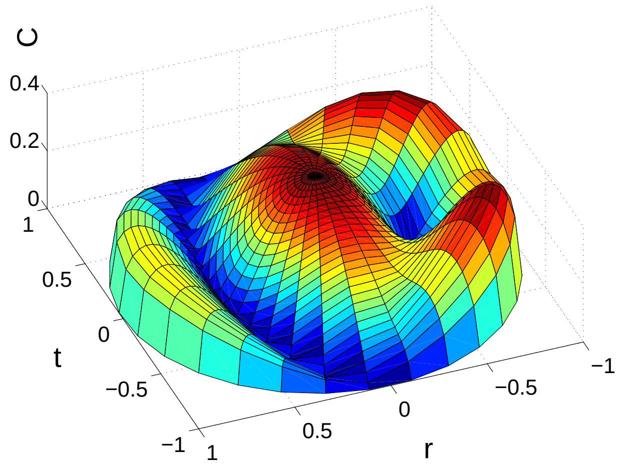

FIG. 2: The concurrence for a pure state of three bosons $|\psi\rangle$ as in Eq. (18) when $\phi = 0$.

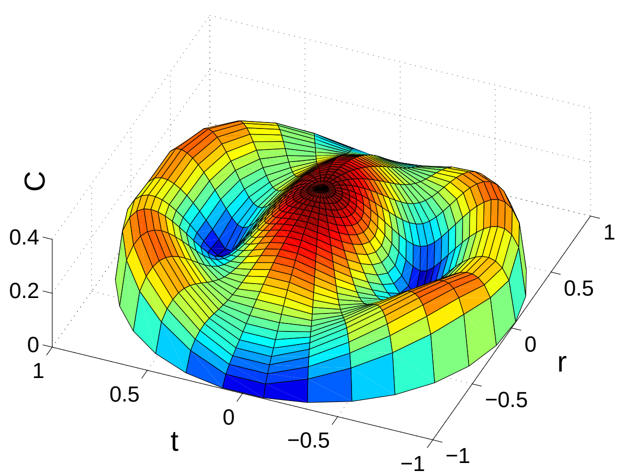

FIG. 3: The concurrence for a pure state of three bosons $|\psi\rangle$ as in Eq. (18) when $\phi = \pi/2$. Note it is symmetric with respect to $t \to -t$.

When $t = 0$, i.e., for a W type entangled state, we find $\mathcal{C} = (\sqrt{6} - \sqrt{2})s^2$ and $\tau = 0$. In this case, the pairwise entanglement increases with the module of s, but there exists no ternary entanglement. When $r = 0$, i.e., for a GHZ type entangled state, we find

$$\mathcal{C} = \sqrt{2}\left|s(\sqrt{3 - 8s^2} - 1)\right|, \tag{21}$$

$$\tau = 16\left|ts^3\right|. \tag{22}$$

In this case, the concurrence $\mathcal{C}$ vanishes for $s = 0$ or $s = 1/2$, where no pairwise entanglement exists. When $s = \sqrt{3}/3$, the concurrence takes the maximum $\mathcal{C} = (\sqrt{6} - \sqrt{2})/3$. Another interesting feature is that there is also a local maximum at $s = \sqrt{6}/8$ with a concurrence $\mathcal{C} = \sqrt{3}/8$. The concurrences for these two special cases of $r = 0$ and $t = 0$ are shown in Fig. 1. The concurrence $\mathcal{C}$ for a general pure state is shown in three dimensional graphs, as in Fig. 2 for $\phi = 0$ and in Fig. 3 for $\phi = \pi/2$.

VI. SPIN SQUEEZING

Spin squeezing results from quantum correlations between individual atomic spins [20, 21]. Recent theoretical investigations have uncovered that spin squeezing is a sufficient but not necessary condition for quantum entanglement [3, 22, 23, 24, 25, 26, 27]. This has led some effort to suggest using the spin squeezing parameter as a multi-atomic entanglement measure [22], as has been fully demonstrated in a two-qubit system [23]. Wang and Sanders [24] illustrated a quantitative relationship between the squeezing parameter and the concurrence for the even and odd (multiple atom spin) states, and have further shown that spin squeezing implies pairwise entanglement for an arbitrary symmetric multi-qubit state [24].

In this section, we investigate the relationship between squeezing parameter and the pairwise concurrence entanglement measures for an arbitrary pure state of three two-state bosons. We start from the standard form Eq. (18) and define the total "pseudo-spin" for three bosons as $\vec{S} = (\vec{\sigma}_1 + \vec{\sigma}_2 + \vec{\sigma}_3)/2$, a direct calculation then gives

$$\langle \vec{S} \rangle = \left[3rs\cos\phi, 3rs\sin\phi, \frac{3}{2}(r^2 + s^2 - t^2)\right]. \tag{23}$$

It is easy to check that the symmetric three two-state boson space consists only of the maximum total spin space satisfying $S^2 = (3/2)(3/2+1)\hbar^2$, which implies a geometric Bloch sphere representation also for the total spin of three two state bosons. We define the unit vector $\hat{z} \propto \langle \vec{S} \rangle$ and choose a cartesian coordinate system with $\hat{x} = (\sin\phi, -\cos\phi, 0)$, $\hat{y} \propto [\cos\phi(r^2 + s^2 - t^2), \sin\phi(r^2 + s^2 - t^2), -2rs]$. This leads to the arbitrary transverse spin direction being $\vec{n}_\perp = \hat{x}\cos(\theta) + \hat{y}\sin(\theta)$ and $S_\perp = \vec{S}\cdot\vec{n}_\perp$. The squeezing parameter ξ is defined by

$$\xi = \frac{4}{3}(\Delta S_\perp)_{\min}. \tag{24}$$

After some tedious calculations, we find

$$\frac{4}{3}(\Delta S_\perp) = A\cos^2\theta + B\cos\theta\sin\theta + C, \tag{25}$$

with expressions for A, B, and C given in Appendix C, and

$$\frac{1}{u} = \sqrt{(r^2 + s^2 - t^2)^2 + 4r^2s^2}. \tag{26}$$

It is reasonably easy to find the minimum of Eq. (25) since it is a simple trigonometric function of the form $(A\cos 2\theta + B\sin 2\theta)/2 + (A/2 + C)$, whose minimum can be found in terms of 2θ and the signs of A and B.

We now discuss three important cases for ξ for an arbitrary pure state of form (18). First, when $s = 0$, we have $|\psi\rangle = r|000\rangle + t|111\rangle$. In this case, we find that ξ is always 1, independent of the values of r and t. This means that these kind of entangled states are never spin-squeezed.

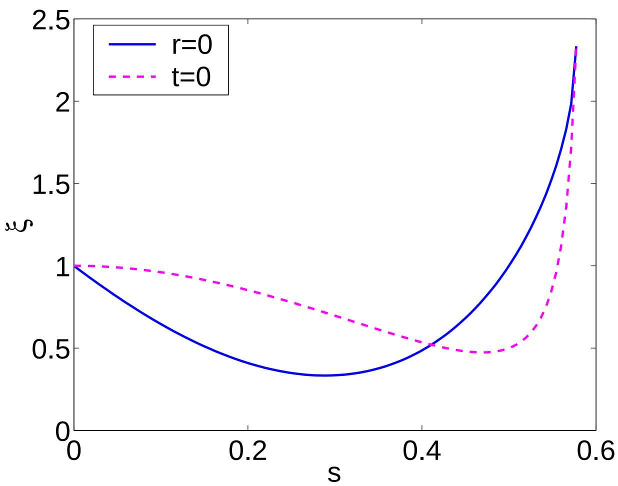

FIG. 4: The squeezing parameter ξ for a pure state of three bosons $|\psi\rangle$ as in Eq. (18) with r=0 (blue solid line) or t=0 (magenta dashed line).

Second, when $t = 0$, we find

$$\xi = \frac{1 - 4s^2 + 16s^6}{1 - 8s^4}, \tag{27}$$

which has one minimum at $s \simeq 0.4694$ with a squeezing parameter $\xi \simeq 0.4738$. The dependence of ξ as a function of s is shown in Fig. 4 in dashed line, where s varies from 0 to $1/\sqrt{3}$. This result is independent of the value of ϕ.

Third when $r = 0$, we find

$$\xi = 1 + 4s^2 - 4\sqrt{s^2 - 3s^4}, \tag{28}$$

which gives $\xi = 1$ for $s = 0$ and $s = 1/2$. Thus there exists no squeezing in these two states. The maximum value of the squeezing parameter is $\xi = 7/3$ in this case, corresponding to $s = \sqrt{3}/3$, i.e. a W state. The minimum value of the squeezing parameter is $\xi = 1/3$ at $s = \sqrt{3}/6$. The squeezing parameter ξ as a function of s is plotted in Fig. 5, independent of ϕ in this case.

Finally, we use two three-dimensional figures to illustrate the squeezing parameter ξ as a function of r and t in Figs. 5 and 6 with both r and t varying from 0 to 1. We have set $\phi = 0$ for Fig. 5 and $\phi = \pi/2$ for Fig. 6.

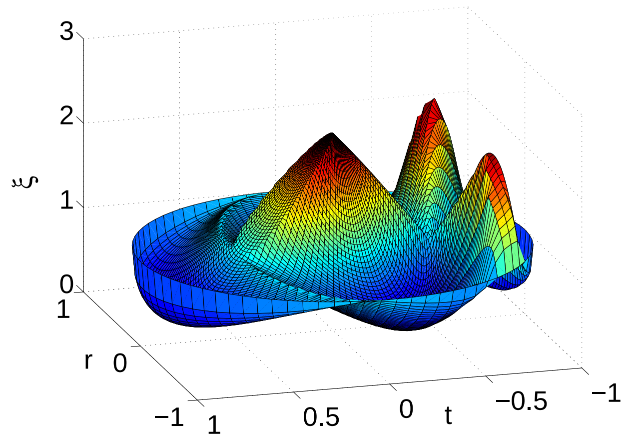

FIG. 5: The squeezing parameter ξ for a pure state of three bosons $|\psi\rangle$ as in Eq. (18) with $\phi = 0$.

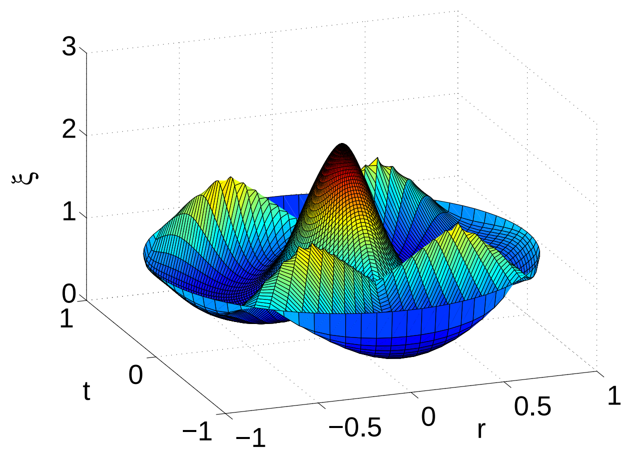

FIG. 6: The squeezing parameter ξ for a pure state of three bosons $|\psi\rangle$ as in Eq. (18) with $\phi = \pi/2$.

VII. CONCLUSION

To clarify the relationship between spin squeezing and pairwise entanglement, let us pay some attention to Figs. 1 and 4. For states of $t = 0$, i.e., W-type entangled states, the concurrence $\mathcal{C}$ is a monotonically increasing quantity with parameter s, while there exists a minimum of the squeezing parameter ξ. Thus, we conclude that for W-type states the spin squeezing is drastically different from the pairwise quantum entanglement. For states of $r = 0$, i.e., GHZ tye-entangled states, we find several common features between pairwise entanglement and spin squeezing: when $s = 0$ and $s = 1/2$, neither pairwise entanglement nor spin squeezing exists. For $s = \sqrt{3}/3$, these two quantities attain the maximum. There also exists another extreme point in these two quantities. However, we note that the parameters corresponding to these

two extreme points are not the same. When $s = \sqrt{6}/8$, the concurrence takes a local maximum value, while the spin squeezing parameter takes a minimum value when $s = \sqrt{3}/6$. This different dependence on the parameter s shows that also for GHZ-type entangled states, the spin squeezing parameter cannot be regarded as a measure of pairwise entanglement. It is worthy to point out that on this point our result is consistent with several previous works [23, 24, 26]. Although for a collection of special states, there might exists a quantitative relationship between pairwise entanglement and spin squeezing, these two properties in general refers to different aspects of multi-party quantum correlation, and are not simply related to each other.

In summary, we have obtained the canonical form of an arbitrary pure state for three two-state bosons. Based on this form, we have classified the entanglement of three identical bosons in two modes into two types, GHZ and W types, analogues to the case of three distinguishable qubits [16]. We have completely studied two important entanglement measures, the concurrence $\mathcal{C}$ and the triple entanglement measure τ, and have also investigated the spin squeezing property of our system by directly computing the spin squeezing parameter ξ. Our results demonstrate that even for pure states of a system of three bosons in two modes, the spin squeezing parameter ξ cannot be regarded as an entanglement measure, in contrast to a system of two particles.

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APPENDIX A: PROOF OF PROPOSITION 1

Proof: For the coefficient of $|000\rangle$ we prove that the solution to the equation

$$a\alpha^3 - d\beta^{*3} - b\beta^*\alpha^2 + c\alpha\beta^{*2} = 0, \tag{A1}$$

does exist. Without loss of generality, we assume $\alpha \neq 0$. Divide the above equation by α^3, we get

$$a - d\left(\frac{\beta^*}{\alpha}\right)^3 - b\left(\frac{\beta^*}{\alpha}\right) + c\left(\frac{\beta^*}{\alpha}\right)^2 = 0. \tag{A2}$$

Of course the solution to Eq. (A2) exits for the variable β^*/α from the fundamental theorem of algebra. Similarly we can prove that the coefficient of $|111\rangle$ can be eliminated by the single particle transformation Eq. (10).

Now we consider the equation

$$\begin{aligned} &a\alpha\beta^2 - d\alpha^*\beta^{*2} - b\beta^*\beta^2 + c\alpha\alpha^{*2} \\ &+2b\alpha\alpha^*\beta - 2c\beta\beta^*\alpha^* = 0. \end{aligned} \tag{A3}$$

Without loss of generality, we assume $\beta \neq 0$. Let $z = \alpha/\beta^*$ and divide Eq. (A3) by $\beta^*\beta^2$, we get

$$az - dz^{*2} - b + czz^{*2} + 2bzz^* - 2cz^* = 0. \tag{A4}$$

Take its complex conjugation, we obtain

$$a^*z^* - d^*z^2 - b^* + c^*z^2z^* + 2b^*zz^* - 2c^*z = 0. \tag{A5}$$

After eliminating variable z^* from Eqs. (A4) and (A5), we are left with a fifth order polynomial equation for the complex variable z. According to the fundamental theorem of Algebra, there exists at least one solution of equation (A4).

A similar procedure can be applied to the coefficient of state $|100\rangle$. Therefore we complete our proof of proposition 1.

APPENDIX B: PROOF OF PROPOSITION 2

Proof:

When $r \neq 0$, $t \neq 0$, and $s = 0$, the standard form itself is just the GHZ type entanglement.

When $r = 0$, $t \neq 0$, and $s \neq 0$, we can choose $|\alpha\rangle = -w|0\rangle + s|1\rangle/2w^2$ and $|\beta\rangle = w|0\rangle + s|1\rangle/2w^2$, where w satisfies $w^6 = s^3/4t$. Thus, we get GHZ type entanglement.

When $r \neq 0$, $t \neq 0$, and $s \neq 0$, we choose $|\alpha\rangle = a|0\rangle + b|1\rangle$ and $|\beta\rangle = c|0\rangle + d|1\rangle$, where $a = t^2s^2r/[(tr^2 + 4s^3)(-t+2u^3)v^2]$, $b = v$, $c = ru(u^3-t)/s(2u^3-t)$, $d = u$. And u satisfies $(tr^2+4s^3)u^6+(-t^2r^2-4ts^3)u^3+t^2s^3 = 0$, v satisfies $v^3 + u^3 - t = 0$. Thus We will get GHZ type entanglement.

When $r \neq 0, t = 0, s \neq 0$, the standard form itself is just the W type entanglement.

When $r \neq 0, t = 0, s \neq 0$, we choose $|\alpha\rangle = r|0\rangle/3+s|1\rangle$, $|\beta\rangle = |0\rangle$, and it becomes the W type.

This completes our proof.

APPENDIX C: EXPRESSIONS FOR A, B, AND C

$$\begin{aligned} A &= 128u^2s^3r^2t\cos^3\phi - 96u^2s^3r^2t\cos\phi - 40u^2s^4r^2 \\ &\quad -64u^2st^3r^2\cos^3\phi - 24u^2st^5\cos\phi + 48u^2s^3t^3\cos\phi \\ &\quad -24u^2s^5t\cos\phi - 64u^2s^3t^3\cos^3\phi + 32u^2s^5t\cos^3\phi \\ &\quad +32u^2st^5\cos^3\phi + 24u^2s^2r^2t^2 + 48u^2st^3r^2\cos\phi \\ &\quad -24u^2sr^4t\cos\phi + 32u^2sr^4t\cos^3\phi - 8u^2s^2r^4, && \text{(C1)} \\ B &= -32ut^3s\cos^2\phi\sin\phi + 32us^3t\cos^2\phi\sin\phi \\ &\quad +32ur^2st\cos^2\phi\sin\phi - 8ur^2st\sin\phi \\ &\quad -8us^3t\sin\phi + 8ut^3s\sin\phi, && \text{(C2)} \\ C &= 1 + 4s^2 + 12st\cos\phi - 16st\cos^3\phi, && \text{(C3)} \end{aligned}$$

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